Proof Given $\sum_{n=1}^{\infty}|b_n|$ is convergent.

$\therefore$ let $\sum_{n=1}^{\infty}|b_n| = M$

given $|a_n| \le |b_n|$ for $n \ge N$

if $s_n = |a_1|+|a_2|+ \cdots +|a_N|+|a_{N+1}|+\cdots+|a_n|$

$s_n \le |a_1|+|a_2|+\cdots+|a_N|+|b_{N+1}|+\cdots+|b_n|$

~~s_n~~ $s_n \le |a_1|+|a_2|+\cdots+|a_N|+M$

$\Rightarrow$ sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above sequence.

Also $s_1 = |a_1|$, $s_2 = |a_1|+|a_2|$ $\cdots$ $s_n = |a_1|+|a_2|+\cdots+|a_n|$

$\Rightarrow$ $s_1 \le s_2 \le \cdots s_n \le s_{n+1} \le \cdots$

seq $\{s_n\}_{n=1}^{\infty}$ increasing sequence

we know that increasing sequence and bounded above is convergent.

$\therefore$ $\{s_n\}_{n=1}^{\infty}$ is convergent sequence

$\Rightarrow$ $\sum_{n=1}^{\infty}|a_n|$ is convergent sequence.

Comparison Test

Theorem 3.6C If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty}|a_n| = \infty$, then $\sum_{n=1}^{\infty}|b_n| = \infty$.

[That is, if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty}|a_n| = \infty$, then $\sum_{n=1}^{\infty}|b_n| = \infty$].

$$|a_n| \leq |b_n| \quad \forall n \geq N$$

$$\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |b_n|$$

$$\sum_{n=\infty}^{\infty} |a_n| = \infty$$

$$\Rightarrow \sum_{n=\infty}^{\infty} |b_n| \geq \infty$$

$$\Rightarrow \sum |b_n| = \infty$$

3.6D Theorem

a) If $\sum_{n=1}^{\infty} b_n$ converges absolutely and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

b) If $\sum_{n=1}^{\infty} |a_n| = \infty$ and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists, then $\sum_{n=1}^{\infty} |b_n| = \infty$.

Proof (a) Given $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists

ie) sequence $\left\{\frac{|a_n|}{|b_n|}\right\}_{n=1}^{\infty}$ is convergent.

We ~~we~~ know that every convergent sequence is bounded

$\therefore$ $\exists$ $M > 0$ such that

$$\frac{|a_n|}{|b_n|} \leq M \quad \forall n \in I$$

$$\Rightarrow |a_n| \leq M|b_n| \quad \forall n \in I$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is dominated by $M \sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty} b_n$ converges absolutely

$\Rightarrow M \sum_{n=1}^{\infty} b_n$ converges absolutely

$\therefore$ By comparison test.

$\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof (b)

$$|a_n| \le M |b_n|$$

$$\Rightarrow \frac{1}{M} |a_n| \le |b_n|$$

$\sum_{n=1}^{\infty} |b_n|$ dominates $\frac{1}{M} \sum_{n=1}^{\infty} |a_n|$

$$\sum_{n=1}^{\infty} |a_n| = \infty \Rightarrow \frac{1}{M} \sum_{n=1}^{\infty} |a_n| = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} |b_n| = \infty$ by comparison test.

Theorem 3.6E **Ratio Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero real numbers and

let $a = \liminf_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$, $A = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

(so that $a \le A$). Then

a) If $A<1$, then $\sum_{n=1}^{\infty}|a_n| < \infty$;

b) If $a>1$ then $\sum_{n=1}^{\infty} a_n$ diverges;

c) If $a \le 1 \le A$, then the test fails.

Proof(a) If $A<1$ Choose any B such that

$A < B < 1$ and $B = A + \epsilon$ some $\epsilon > 0$

since $A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

there exists $N \in I$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \le A + \epsilon \quad \forall\, n \ge N$$

then $$\left|\frac{a_{n+1}}{a_n}\right| \le B \quad \forall\, n \ge N$$

$$\Rightarrow \quad \frac{|a_{N+1}|}{|a_N|} \le B \quad \text{—— } (1)$$

$$\frac{|a_{N+2}|}{|a_{N+1}|} \le B \quad \text{—— } (2)$$

$$\vdots$$

$$\frac{|a_{N+k}|}{|a_{N+k-1}|} \le B \quad \text{—— } (k)$$

$(1) \times (2) \times (3) \ldots \times (k)$

$$\frac{|a_{N+k}|}{|a_N|} \le B^k \qquad k = 0, 1, 2, \ldots$$

$$|a_{N+k}| \le |a_N| B^k$$

$$\sum_{k=0}^{\infty} |a_{N+k}| \le |a_N| \sum_{k=0}^{\infty} B^k \quad \text{——} \; (*)$$

But $B<1 \Rightarrow \sum_{k=0}^{\infty} B^k$ is convergent.

$\Rightarrow |a_N| \sum_{k=0}^{\infty} B^k$ is convergent.

using $(*)$ and Comparison test

$\sum_{k=0}^{\infty} |a_{N+k}|$ is convergent.

$\sum_{k=0}^{\infty} |a_{N+k}| = |a_N| + |a_{N+1}| + \cdots$ is convergent

$\Rightarrow |a_1| + |a_2| + \cdots + |a_{N-1}| + |a_N| + |a_{N+1}| + \cdots$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ is convergent.

Proof (b) If $a > 1$

since $a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$\Rightarrow \frac{|a_{n+1}|}{|a_n|} > 1 \quad \forall n \ge N$$

$$|a_{n+1}| > |a_n| \quad \forall n \ge N$$

$$\Rightarrow |a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$$

$$\Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Proof (c) To illustrate conclusion (c) consider first $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$

Here $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

$\Rightarrow$ $a = 1 = A$ The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ $a = A = 1$ But the series

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Root test.

Theorem 3.6F If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = A$ then the series of real numbers $\sum_{n=1}^{\infty} a_n$ (a) converges absolutely if $A < 1$ (b) diverges if $A > 1$ If $A = 1$ the test fails.

Proof If $A < 1$ choose B so that $A < B < 1$ where $B = A + \epsilon < 1$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = A$$

$$\Rightarrow \quad \sqrt[n]{|a_n|} < A + \epsilon \quad \forall n \geq N$$

$$\sqrt[n]{|a_n|} < B \quad \forall n \geq N.$$

$$|a_n| < B^n \quad n \geq N$$

$\sum_{n=1}^{\infty} |a_n|$ is dominated by $\sum_{n=1}^{\infty} B^n$

But $B<1 \Rightarrow \sum_{n=1}^{\infty} B^{n}$ Convergent.

$\therefore$ By Comparison Test

$\sum_{n=1}^{\infty} |a_n|$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent absolutely.

Proof(b) ~~A~~ If $A>1$

$\sqrt[n]{|a_n|} > 1$

$|a_n| > 1 \quad \forall\, n \geq N$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n$ is diverges

If $A=1$

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} e^{(-\log n/n)} = e^{0} = 1 = A$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}\right)^2} = 1 = A.$$

But the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is converges $\therefore$ If $A=1$ This test fails.